$\boxed{① (a)}$ We want to solve

$$y' - 2y = 1$$

on $I = (-\infty, \infty)$.

Let $A(x) = \int -2\,dx = -2x$

Multiply by $e^{A(x)} = e^{-2x}$ to get

$$e^{-2x} y' - 2e^{-2x} y = e^{-2x}$$

This gives

$$\left(e^{-2x} y\right)' = e^{-2x}$$

Integrating with respect to x gives

$$e^{-2x} y = \int e^{-2x}\,dx$$

So, $e^{-2x} \cdot y = -\frac{1}{2} e^{-2x} + C$

Thus, $y = -\frac{1}{2} \underbrace{e^{2x} e^{-2x}}_{} + Ce^{2x}$

$e^{2x} \cdot e^{-2x} = e^{2x-2x}$
$= e^{0} = 1$

Answer: $\boxed{y = -\frac{1}{2} + Ce^{2x}}$

Check answer:

$y = -\frac{1}{2} + Ce^{2x}$
$y' = 2Ce^{2x}$

$y' - 2y = 2Ce^{2x} + 1 - 2Ce^{2x} = 1$

1 (b)

We want to solve
$$y' + 2xy = x$$
on $I = (-\infty, \infty)$

Let $A(x) = \int 2x\,dx = x^2$

Multiply by $e^{A(x)} = e^{x^2}$ to get
$$e^{x^2} y' + 2xe^{x^2} y = xe^{x^2}$$

So,
$$(e^{x^2} \cdot y)' = xe^{x^2}$$

Thus, by integrating with respect to x we get
$$e^{x^2} \cdot y = \int xe^{x^2}\,dx$$

Note that
$$\int xe^{x^2}\,dx = \int \tfrac{1}{2} e^{u}\,du = \tfrac{1}{2}e^{u} + C = \tfrac{1}{2}e^{x^2} + C$$

$u = x^2$
$du = 2x\,dx$
$\frac{1}{2}du = x\,dx$

Thus,
$$e^{x^2} \cdot y = \tfrac{1}{2}e^{x^2} + C$$

So,

$$y = \frac{1}{2}\underbrace{e^{-x^2} \cdot e^{x^2}}_{e^{-x^2} \cdot e^{x^2} = e^{-x^2+x^2} = e^0 = 1} + C \cdot e^{-x^2}$$

Thus,

$$\boxed{y = \frac{1}{2} + Ce^{-x^2}}$$

Answer

Check answer:

$$y = \frac{1}{2} + Ce^{-x^2}$$

$$y' = -2xCe^{-x^2}$$

$$y' + 2xy = -2Cxe^{-x^2} + x + 2Cxe^{-x^2} = x$$

① (c) We want to solve

$$\frac{dy}{dx} + e^x y = 3e^x$$

on $I = (-\infty, \infty)$.

Let $A(x) = \int e^x \, dx = e^x$.

Multiply both sides by $e^{A(x)} = e^{e^x}$ to get

$$e^{e^x} \cdot \frac{dy}{dx} + e^x e^{e^x} \cdot y = 3e^x e^{e^x}$$

Thus,

$$\left(e^{e^x} \cdot \frac{dy}{dx} \right)' = 3e^x e^{e^x}$$

Integrating with respect to x gives

$$e^{e^x} \cdot \frac{dy}{dx} = \int 3e^x \cdot e^{e^x} \, dx$$

Note that

$$\int 3e^x \cdot e^{e^x} \, dx = 3 \int e^u \, du = 3e^u + C$$

$$= 3e^{e^x} + C$$

$$\boxed{\begin{gathered} u = e^x \\ du = e^x \, dx \end{gathered}}$$

Thus,

$$e^{e^x} \cdot \frac{dy}{dx} = 3e^{e^x} + C$$

So,

$$\frac{dy}{dx} = 3\underbrace{e^{-e^x} \cdot e^{e^x}}_{e^{-e^x} \cdot e^{e^x} = e^{-e^x + e^x} = e^0 = 1} + Ce^{-e^x}$$

Thus,

$$\boxed{\frac{dy}{dx} = 3 + Ce^{-e^x}}$$

Answer

$\boxed{\text{1 (d)}}$ We want to solve

$$\frac{dy}{dx} + 2xy = xe^{-x^2}$$

on $I = (-\infty, \infty)$

Let $A(x) = \int 2x\,dx = x^2$

Multiply both sides by $e^{A(x)} = e^{x^2}$ to get

$$e^{x^2} \cdot \frac{dy}{dx} + 2xe^{x^2} y = x\underbrace{e^{x^2} e^{-x^2}}$$

$e^{x^2} e^{-x^2} = e^{x^2 - x^2} = e^0 = 1$

Thus,

$$\left(e^{x^2} \cdot y\right)' = x$$

Integrating with respect to x gives

$$e^{x^2} \cdot y = \int x\,dx$$

$$= \frac{x^2}{2} + C$$

Thus,

$$\boxed{y = \frac{1}{2}x^2 e^{-x^2} + Ce^{-x^2}}$$

Answer

1 (e) We want to solve

$$y' = e^{3x} + \sin(x)$$

on $I = (-\infty, -\infty)$.

This one we can just integrate both sides since there is no y term.

So we get

$$y = \int e^{3x}\,dx + \int \sin(x)\,dx$$

Thus,

$$\boxed{y = \frac{1}{3}e^{3x} - \cos(x) + C}$$

1 (f) We want to solve

$$y' - (\tan(x)) \cdot y = e^{\sin(x)}$$

on $I = (0, \frac{\pi}{2})$.

Let $A(x) = \int -\tan(x)\,dx$

$$= -\ln|\sec(x)|$$

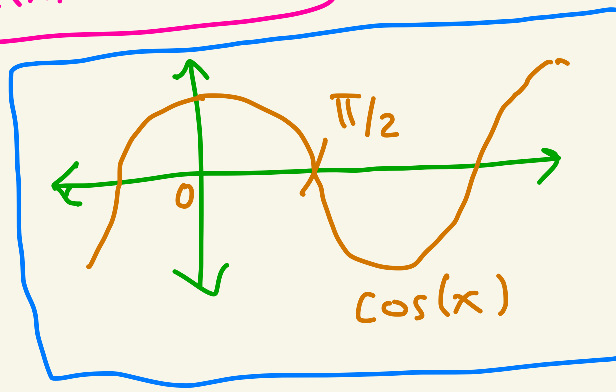
$-\ln(A) = \ln\left(\frac{1}{A}\right)$

$$= \ln\left(\frac{1}{|\sec(x)|}\right)$$

$\frac{1}{\sec(x)} = \cos(x)$

$$= \ln|\cos(x)|$$

Note that on $I = (0, \pi/2)$ we have that $\cos(x) > 0$



Thus, on $I = (0, \pi/2)$ we have

$$A(x) = \ln|\cos(x)| = \ln(\cos(x))$$

$|\cos(x)| = \cos(x)$ when $\cos(x) > 0$

Multiply both sides of the ODE by

$e^{\ln(t)} = t$

$$e^{A(x)} = e^{\ln(\cos(x))} = \cos(x)$$

to get

$$\cos(x)\,y' - \cos(x)\underbrace{\tan(x)}_{\frac{\sin(x)}{\cos(x)}}\,y = \cos(x)\,e^{\sin(x)}$$

This simplifies to

$$\cos(x)\,y' - \sin(x)\,y = \cos(x)\,e^{\sin(x)}$$

So we get

$$\left(\cos(x)\cdot y\right)' = \cos(x)\,e^{\sin(x)}$$

Integrating both sides with respect to x gives

$$\cos(x)\cdot y = \int \cos(x)\,e^{\sin(x)}\,dx$$

And

$$\int \cos(x)\,e^{\sin(x)}\,dx = \int e^{u}\,du = e^{u} + C = e^{\sin(x)} + C$$

$u = \sin(x)$
$du = \cos(x)\,dx$

Thus,

$$\cos(x)\cdot y = e^{\sin(x)} + C$$

So,

$$\boxed{y = \frac{1}{\cos(x)}\,e^{\sin(x)} + \frac{C}{\cos(x)} = \sec(x)\,e^{\sin(x)} + C\sec(x)}$$

Answer

② (a) We want to solve

$$3\frac{dy}{dx} + y = 2e^{-x}$$

on $I = (-\infty, \infty)$.

Divide by 3 to put the ODE into standardized form. We get

$$\frac{dy}{dx} + \frac{1}{3}y = \frac{2}{3}e^{-x}$$

Let $A(x) = \int \frac{1}{3}\,dx = \frac{1}{3}x$.

Multiply both sides by $e^{A(x)} = e^{\frac{1}{3}x}$ to get

$$e^{\frac{1}{3}x} \cdot \frac{dy}{dx} + \frac{1}{3}e^{\frac{1}{3}x}y = \frac{2}{3}\underbrace{e^{\frac{1}{3}x}e^{-x}}$$

$$e^{\frac{1}{3}x} \cdot e^{-x} = e^{\frac{1}{3}x - x} = e^{-\frac{2}{3}x}$$

So,

$$\left(e^{\frac{1}{3}x} \cdot y\right)' = \frac{2}{3}e^{-\frac{2}{3}x}$$

Thus,

$$e^{\frac{1}{3}x} \cdot y = \underbrace{\int \frac{2}{3}e^{-\frac{2}{3}x}\,dx}$$

$$\int \frac{2}{3}e^{-\frac{2}{3}x}\,dx = \frac{2}{3} \cdot \left(-\frac{3}{2}e^{-\frac{2}{3}x}\right) + C$$

$$= -e^{(-2/3)x} + C$$

Thus,

$$e^{\frac{1}{3}x} \cdot y = -e^{-\frac{2}{3}x} + c$$

So,

$$y = -\underbrace{e^{-\frac{1}{3}x} e^{-\frac{2}{3}x}} + c e^{-\frac{1}{3}x}$$

$$e^{-\frac{1}{3}x} e^{-\frac{2}{3}x} = e^{-\frac{1}{3}x - \frac{2}{3}x} = e^{-x}$$

Thus,

$$\boxed{y = -e^{-x} + c e^{-\frac{1}{3}x}}$$

Answer

2 (b) We want to solve

$$xy' + y = 3x^3 - 1$$

on $I = (0, \infty)$.

Divide by x to standardize the equation.

We get

$$y' + \frac{1}{x} y = 3x^2 - \frac{1}{x}$$

Let

$$A(x) = \int \frac{1}{x}\,dx = \ln|x| = \ln(x)$$

$x > 0$
on $I = (0, \infty)$
so, $|x| = x$

Multiply both sides by

$$e^{A(x)} = e^{\ln(x)} = x$$

to get

$$xy' + y = 3x^3 - 1$$

This becomes

$$(xy)' = 3x^3 - 1$$

Thus, by integrating both sides with respect to x we get

$$xy = \int (3x^3 - 1)\,dx$$

So,

$$xy = \frac{3}{4}x^4 - x + C$$

Thus,

$$\boxed{y = \frac{3}{4}x^3 - 1 + \frac{C}{x}}$$

Answer

2 (c) We want to solve

$$x^2 y' + x(x+2)y = e^x$$

on $I = (0, \infty)$

First divide by x^2 to put the ODE into standardized form. We get

$$y' + \left(1 + \frac{2}{x}\right)y = x^{-2}e^x$$

Let

$$A(x) = \int \left(1 + \frac{2}{x}\right)dx = x + 2\ln|x| = x + 2\ln(x)$$

since $I = (0, \infty)$ we have $x > 0$ so $|x| = x$

Multiply both sides of the ODE by $e^{A(x)} = e^{x + 2\ln(x)} = e^x e^{2\ln(x)}$

$$= e^x e^{\ln(x^2)} = x^2 e^x$$

$e^{\ln(t)} = t$

to get

$$x^2 e^x y' + x^2 e^x \left(1 + \frac{2}{x}\right)y = x^2 e^x x^{-2} e^x$$

This simplifies to

$$x^2 e^x y' + (x^2 + 2x)e^x y = e^{2x}$$

We get

$$(x^2e^xy)' = e^{2x}$$

Check:

$$(x^2e^xy)'$$
$$= (x^2e^x)'y + x^2e^xy'$$
$$= (2xe^x + x^2e^x)y + x^2e^xy'$$
$$= x^2e^xy' + (x^2+2x)e^xy$$

Integrating with respect to x gives

$$x^2e^xy = \frac{1}{2}e^{2x} + C$$

So,

$$\boxed{\begin{aligned} y &= \frac{1}{2}x^{-2}e^{-x}e^{2x} + Cx^{-2}e^{-x} \\ &= \frac{e^x}{2x^2} + \frac{C}{x^2e^x} \end{aligned}}$$

Answer

$\boxed{②(d)}$ We want to solve

$$(x^2+9)\frac{dy}{dx} + xy = 0$$

on $I = (-\infty, \infty)$

Divide by x^2+9 to put the ODE into a standardized form. We get

$$\frac{dy}{dx} + \frac{x}{x^2+9}\,y = 0$$

Let $A(x) = \int \frac{x}{x^2+9}\,dx = \int \frac{1}{2}\frac{1}{u}\,du = \frac{1}{2}\ln|u|$

$u = x^2+9$
$du = 2x\,dx$
$\frac{1}{2}du = x\,dx$

$= \frac{1}{2}\ln|x^2+9|$

$= \frac{1}{2}\ln(x^2+9)$

$x^2+9>0$ always

Multiply both sides of the ODE by

$$e^{A(x)} = e^{\frac{1}{2}\ln(x^2+9)} = e^{\ln((x^2+9)^{1/2})} = (x^2+9)^{1/2}$$

to get

$$(x^2+9)^{1/2}\cdot\frac{dy}{dx} + (x^2+9)^{1/2}\frac{x}{(x^2+9)}\,y = 0$$

This simplifies to

$$(x^2+9)^{1/2}\frac{dy}{dx} + \frac{x}{(x^2+9)^{1/2}}\,y = 0$$

This becomes

$$\left[(x^2+9)^{1/2}\cdot y\right]' = 0$$

Integrating with respect to x gives

$$(x^2+9)^{1/2}\cdot y = C$$

So,

$$\boxed{y = \frac{C}{(x^2+9)^{1/2}} = \frac{C}{\sqrt{x^2+9}}}$$

3. We saw in the previous problems that the general solution to

$$(x^2+9)\frac{dy}{dx} + xy = 0$$

on $I = (-\infty, \infty)$ is

$$y = \frac{c}{\sqrt{x^2+9}}$$

We want the solution to also satisfy $y(0)=1$. plug in $x=0$ to get

$$1 = y(0) = \frac{c}{\sqrt{0^2+9}}$$

So, $1 = \frac{c}{3}$.

Thus, $c = 3$.

So the solution is

$$y = \frac{3}{\sqrt{x^2+9}}$$

4 We want to solve

$$\frac{dy}{dx} + 2xy = x \ , \ y(0) = -3$$

on $I = (-\infty, \infty)$

First we must find the general solution to

$$\frac{dy}{dx} + 2xy = x$$

Let

$$A(x) = \int 2x\,dx = x^2$$

multiply both sides by $e^{A(x)} = e^{x^2}$ to get

$$e^{x^2} \cdot \frac{dy}{dx} + 2xe^{x^2} y = xe^{x^2}$$

This gives

$$\left(e^{x^2} \cdot y\right)' = xe^{x^2}$$

Integrating both sides with respect to x gives

$$e^{x^2} \cdot y = \underbrace{\int xe^{x^2}\,dx}$$

$$\int xe^{x^2}\,dx = \int \tfrac{1}{2} e^u\,du = \tfrac{1}{2} e^u = \tfrac{1}{2} e^{x^2} + C$$

$u = x^2$
$du = 2x\,dx$
$\tfrac{1}{2}\,du = x\,dx$

So,

$$e^{x^2}\cdot y = \frac{1}{2}e^{x^2} + C$$

Thus,

$$y = \frac{1}{2}\underbrace{e^{-x^2}\cdot e^{x^2}} + Ce^{-x^2}$$

$$\boxed{e^{-x^2}\cdot e^{x^2} = e^{-x^2+x^2} = e^0 = 1}$$

So,

$$y = \frac{1}{2} + Ce^{-x^2}$$

We want $y(0) = -3$. Plugging this into the above we get

$$-3 = y(0) = \frac{1}{2} + Ce^{-(0)^2}$$

So,

$$-3 = \frac{1}{2} + Ce^0 = \frac{1}{2} + C$$

Thus,

$$C = -3 - \frac{1}{2} = -\frac{7}{2}.$$

So,

$$\boxed{y = \frac{1}{2} - \frac{7}{2}e^{-x^2}}$$

5. We want to solve

$$xy' + y = 2x, \quad y(1) = 0$$

on $I = (0, \infty)$

First put the equation into a standardized form by dividing through by x to get

$$y' + \frac{1}{x} y = 2$$

Let

$$A(x) = \int \frac{1}{x} dx = \ln|x| = \ln(x)$$

$x > 0$ since $I = (0, \infty)$

multiply both sides by

$$e^{A(x)} = e^{\ln(x)} = x$$

to get

$$xy' + y = 2x$$

The same as where we started! we didn't need the $e^{A(x)}$ it turns out.

This gives

$$(x \cdot y)' = 2x$$

Integrating with respect to x gives

$$x \cdot y = \int 2x \, dx$$

So,

$$x \cdot y = x^2 + c$$

Thus,

$$y = x + \frac{c}{x}$$

We want $y(1) = 0$. Plugging this in gives

$$0 = y(1) = 1 + \frac{c}{1}$$

So,

$$0 = 1 + c$$

Thus,

$$c = -1.$$

Therefore, the solution is

$$\boxed{y = x - \frac{1}{x}}$$